ENDOGENOUS POPULATION IN A NEOCLASSICAL GROWTH MODEL WITH WEALTH AND TIME VALUES

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Abstract: The purpose of this study is to introduce endogenous population growth and portfolio equilibrium between physical wealth and gold into the Solow growth model. The study deals with dynamic interdependence between the birth rate, the mortality rate, the population growth, wealth accumulation, portfolio equilibrium between physical wealth and gold, and time distribution between work, leisure and children caring. The study studies the role of human capital, technological and preference changes on the birth and mortality rates, portfolio choice, and time distribution. We define a dynamic model and provide a computational procedure to demonstrate the motion of the economic system. As a genuine dynamic analysis is too complicated, we simulate the model. We show the motion of the economic growth and population change and identify the existence of equilibrium point. We also examine the effects of changes on the transitory processes and long term equilibrium point in the propensity to use gold, the propensity to have children, the propensity to save, woman’s propensity to use leisure, woman’s human capital and man’s emotional involvement in children caring.

Keywords: Gold Value, Propensity to Have Children, Birth and Mortality Rates, Gender Difference in Time Distribution.

1. INTRODUCTION

The complexity of portfolio has been increasingly increased in modern times. Households of contemporary economies are characterized of many assets such as housing, land, stocks, precious metals, gold, cashes in different currencies, and savings in different countries in their portfolios. It is obvious that genuine dynamic theories are needed in order to properly analyze portfolio equilibrium dynamics in association with economic growth and economic structural changes. Modern economic theory has only a few mathematical models with multiple assets based on microeconomic foundation. This study proposes a mathematical model to deal with growth with portfolio choice equilibrium with gold and physical capital within a comprehensive framework.

The model is influenced by some traditional theories. The unique feature of this paper is to connect physical wealth and gold value determination with accumulation of wealth and population dynamics. The economic aspects of this paper are strongly influenced by the Solow model and some neoclassical growth models with endogenous population. The Solow model has been generalized and extended in different directions (Azariadis, 1993; Barro and Sala-i-Martin, 1995; Burmeister and Dobell, 1970; Solow, 1956). This study is based on the extended Solow model by Zhang (1993) to model economic growth. This study is to examine gold price dynamics within a neoclassical growth model with gender division of labor and children caring. Dynamics of gold prices are well mentioned but not properly theoretically examined (Barro, 1979; Bordo and Ellson, 1985; Chappell and Dowd, 1997; Dowd and Sampson, 1993).

The model treats population change as endogenous. We are dealing with interaction between economic growth and population change. The issues related to the dynamic interdependence have been examined within different theoretical frameworks since Malthus published his an Essay on the Principle of Population in 1798. Modern economies experience unprecedented population dynamics (such as aging and declining birth and mortality rates in developed economies). This study introduces endogenous birth and mortality rates into the Solow one sector growth model. The basic variables for determining the population growth rate is the birth rate and mortality rate. The two rates may be determined with different factors. Fertility may be influenced by factors such as changes in gender gap in wages (Galor and Weil, 1996).
labor market frictions (Adsera, 2005), and age structure (Hock and Weil, 2012). Barro and Becker (1989) deal with dynamics between endogenous fertility and economic growth in an overlapping generation model. Becker G. et al. (1990) argue that there is a quality-quantity trade-off on children as it costs to bring up children to adulthood and provide them education. In the approaches by Galor and Weil (1999) and Doepke (2004), the quality-quantity trade-off on children has been treated as a factor which affects the transition of economies from a stage of stagnation to perpetual growth. Bosi and Seegmuller (2012) study heterogeneity of households in terms of capital endowments, mortality, and costs per surviving child. According to their analysis, a rise in mortality increases the time cost per surviving child and enhances economic growth. Varvarigos and Zakaria (2013) examine interactions between fertility choice and expenditures on health in the traditional overlapping-generations framework (Bhattacharya and Qiao, 2007; Manuelli and Seshadri, 2009). There are other studies on fertility and economic growth (Acemoglu and Johnson, 2007; Ehrlich and Lui, 1997; Galor, 2012; Kirk, 1996; Strulik, 2008). Many studies are concerned with social and economic determinants of mortality (Balestra and Dottori, 2012; Blackburn and Cipriani, 2005; Chakraborty, 2004; Hazan and Zoabi, 2006; Lancia and Prarolo, 2012; Robinson and Srinivasan, 1997). There is also a large literature of studies on the impact of old-age dependency on fertility through the pension system (Cigno and Rosati, 1996; Wigger, 1999).

Our study is strongly influenced by the literature of the neoclassical growth theory and the literature of population growth and economic development. A unique contribution of this paper is to model population and economic growth with portfolio equilibrium between physical wealth and gold. The paper analyzes the link between wealth growth, economic growth, gold value, gender division of labor, and population growth. The physical capital accumulation is built on the Solow growth model. The birth rate and mortality rate dynamics are influenced by the Haavelmo population model and the Barro-Becker fertility choice model. We synthesize these dynamic mechanisms in a compact framework, applying an alternative utility function proposed by Zhang (1993). The model is actually a synthesis of Zhang’s two models (Zhang, 2015;2016). The paper is organized as follows. Section 2 introduces the basic model with wealth accumulation and population growth. Section 3 simulates the model. Section 4 carries out comparative dynamic analysis with regard to some parameters. Section 5 concludes the study.

2. THE BASIC MODEL

The model is to synthesize the two models proposed by(Zhang, 2015;2016). The economic production is neoclassical. The production sector is based on Solow’s one-sector growth model (Solow, 1956). Like the Solow model, the sector produces one homogeneous commodity for consumption and investment. Physical capital is assumed to depreciate at a constant exponential rate, \( \delta \). We consider a perfectly competitive economy. Factors are inelastically supplied. The available factors are fully utilized. Saving is undertaken only by households. All earnings of firms are distributed in the form of payments to factors of production. Households own assets in terms of physical capital and gold and distribute their disposable incomes to consumption, child bearing, use of gold, and saving. The population of each gender is homogeneous. We assume that each family consists of husband, wife and children. All the families are identical. We use subscripts \( q = 1 \) and \( q = 2 \) to stand for man and woman respectively. We use \( N(t) \) to stand for the population of each gender. We use \( T_q(t) \) to stand for time spent on taking care of children of gender \( q \) and \( \bar{N}(t) \) for the flow of labor services used in time \( t \) for production. The total labor supply \( \bar{N}(t) \) is

\[
\bar{N}(t) = [h_1 T_1(t) + h_2 T_2] N(t), \tag{1}
\]

where \( h_q \) is the level of human capital of gender \( q \). We assume that human capital is constant in this study.

2.1. The Industrial Sector

The production sector uses capital \( K(t) \) and labor \( \bar{N}(t) \) as inputs. We use \( F(t) \) to represent the output level at time \( t \). The production function is

\[
F(t) = AK^\alpha(t) \bar{N}^\beta(t), \quad \alpha, \beta > 0, \quad \alpha + \beta = 1, \tag{2}
\]
where \( A, \alpha, \) and \( \beta \) are parameters. Markets are competitive; thus labor and capital earn their marginal products, and firms earn zero profits. Let the rate of interest and the wage rate per unit of time be denoted, respectively, \( r(t) \) and \( w(t) \). For any individual firm, \( r(t) \) and \( w(t) \) are given at any point in time. The marginal conditions are

\[
r(t) + \delta_t = \frac{\alpha F(t)}{K(t)}, \quad w(t) = \frac{\beta F(t)}{N(t)}, \quad w_q(t) = h_q w(t).
\]

(3)

2.2. Portfolio Equilibrium

Let \( p_g(t) \) stand for the price of gold. For convenience of analysis, it is assumed that gold held by anyone can be “rented” through markets for decoration use. The rent of gold is denoted by \( R_g(t) \). The gold owned by the representative household is assumed to be fully used by the household for decoration or for renting out to other households. Consider now an investor with one unity of money. He can either invest in capital good thereby earning a profit equal to the net own-rate of return \( r(t) \) or invest in gold thereby earning a profit equal to the net own-rate of return \( \frac{R_g(t)}{p_g(t)} \). As we assume capital and gold markets to be at competitive equilibrium at any point in time, the two options yield equal returns, i.e.

\[
\frac{R_g(t)}{p_g(t)} = r(t).
\]

(4)

This equation enables us to determine portfolio choice between gold and (physical) wealth. We make some strict assumptions to build the portfolio equilibrium condition. Equations (4) are established under many strict conditions. For instance, we omit any transaction costs and any time delay for buying and selling. Expectations and risks are complicated issues. Equation (4) also implies perfect information.

2.3. The Current Income, Disposable Income, and Time Constraint

Consumers decide time distribution, consumption level of commodity, amount of gold to use, number of children, and amount of saving. Rather than the Ramsey approach in the traditional optimal growth theory, we use an alternative approach to household proposed by Zhang (1993). We use \( \bar{g}(t) \) to stand for the amount of gold owned by the household. To describe behavior of consumers, we denote per family wealth by \( \bar{k}(t) \), where \( \bar{k}(t) = K(t)/N(t) \). The total value of wealth owned by the household \( a(t) \) is the sum of the two assets’ values

\[
a(t) = \bar{k}(t) + p_g(t) \bar{g}(t).
\]

(5)

Per family current income from the interest payment \( r(t)\bar{k}(t) \), the wage payments and the gold interest income \( R_g(t)\bar{g}(t) \) is

\[
y(t) = r(t)\bar{k}(t) + \left[ h_1 T_1(t) + h_2 T_2(t) \right] w(t) + R_g(t)\bar{g}(t).
\]

We call \( y(t) \) the current income. The total value of wealth that a family can sell to purchase goods and to save is equal to \( a(t) \). Here, we assume that selling and buying wealth can be conducted instantaneously without any transaction cost. The disposable income per family is given by

\[
\hat{y}(t) = y(t) + a(t).
\]

(6)

We consider that the total available time is distributed between work, child caring, and leisure. We denote the leisure time of gender \( q \) by \( \bar{T}_q(t) \). We use \( \bar{T}_q(t) \) to represent time spent on taking care of children of gender \( q \). An adult is faced with the following time constraint
\[ T_q(t) + \tilde{T}_q(t) + \tilde{\tilde{T}}_q(t) = T_0, \]  

where \( T_0 \) is the total available time for leisure, work and children caring.

### 2.4. The Financial and Time Costs of Bringing Up Children

Let \( n(t) \) and \( p_b(t) \) stand for the birth rate and the cost of birth at time. Following Zhang (2015), we assume that children will have the same level of wealth as that of the parent. In addition to the time spent on children, the cost of the parent is given by

\[ p_b(t) = n(t)a(t). \]

Here, we neglect other costs such as purchases of goods and services. In the fertility choice model by Barro and Becker (1989), the cost also includes consumption of goods. We now introduce time spent on children. Becker G. S. (1981) emphasizes costs of the mother’s time on rearing children to adulthood. In this study we assume the following relation between fertility rate and the parent’s time on raising children

\[ \overline{T}_q(t) = \theta_q n(t), \quad \theta_q \geq 0. \]

The specified function form implies positively proportional relation between the time of children parents want more children, they spend more time on child caring. This requirement is strict as child caring tends to exhibit increasing return to scale. For instance, the time per child tends to fall as the family has more children. We require the constant return to scale because this assumption makes the analysis mathematically tractable.

### 2.5. The Budget, Utility Function and Optimal Decision

The household distributes the total available budget between saving, \( s(t) \), consumption of goods, \( c(t) \), use of gold \( \hat{g}(t) \), and bearing children, \( p_b(t) \). The budget constraint is

\[ c(t) + R_s(t) \hat{g}(t) + s(t) + a(t)n(t) = \tilde{y}(t), \]

Substituting (6) and (7) into (10) yields

\[ c(t) + R_s(t) \hat{g}(t) + s(t) + a(t)n(t) + \overline{T}_1(t)w_1(t) + \overline{T}_2(t)w_2(t) + \tilde{T}_1(t)w_1(t) + \tilde{T}_2(t)w_2(t) = \overline{y}(t), \]

where

\[ \overline{y}(t) = (1 + r(t))a(t) + (w_1(t) + w_2(t))\theta_q. \]

The right-hand side is the “potential” income that the family can obtain by spending all the available time on work. The left-hand side is the sum of the consumption cost, the saving, the opportunity cost of bearing children, and opportunity cost of leisure. Insert (10) in (11)

\[ c(t) + R_s(t) \hat{g}(t) + s(t) + \tilde{w}(t)n(t) + \overline{T}_1(t)w_1(t) + \tilde{T}_2(t)w_2(t) = \overline{y}(t), \]

where

\[ \tilde{w}(t) \equiv a(t) + hw(t), \quad h \equiv \theta_1 h_1 + \theta_2 h_2. \]

The variable \( \tilde{w}(t) \) is the opportunity cost of children fostering. Like Barro and Becker (1989), we introduce the number of children into the parents’ utility. The utility is assumed to be dependent on \( c(t) \), \( \hat{g}(t) \), \( s(t) \), \( \overline{T}_q(t) \), and \( n(t) \) as follows
\[ U(t) = c^{\xi_0}(t) \hat{g}^{\gamma}(t) s^{\alpha}(t) \tilde{T}_1^{\sigma_{0q}}(t) \tilde{T}_2^{\sigma_{0q}}(t) n^{\nu}(t), \]

where \( \xi_0 \) is called the propensity to consume, \( \gamma_0 \) the propensity to use gold, \( \lambda_0 \) the propensity to own wealth, \( \sigma_{0q} \) the gender \( q \)'s propensity to use leisure time, and \( \nu_0 \) the propensity to have children. It should be noted that there are many other factors related to number of children that may affect parents’ decision. For instance, Soares (2005) consider parents’ utility dependent on their surviving offsprings, as well as length of each surviving child’s lifespan.

The first-order condition of maximizing \( U(t) \) subject to (12) yields

\[ c(t) = \xi \lambda(t), \quad \hat{g}(t) = \frac{\gamma \lambda(t)}{\hat{R}_t(t)}, \quad s(t) = \lambda \lambda(t), \quad \tilde{T}_q(t) = \frac{\sigma_q \lambda(t)}{\lambda_q(t)}, \quad n(t) = \nu \lambda(t), \]

where

\[ \xi \equiv \rho \xi_0, \quad \gamma_0 \equiv \gamma \rho, \quad \lambda \equiv \rho \lambda_0, \quad \sigma_q \equiv \rho \sigma_{0q}, \quad \nu \equiv \rho \nu_0, \quad \rho \equiv \frac{1}{\xi_0 + \gamma_0 + \lambda_0 + \sigma_{10} + \sigma_{20} + \nu_0}. \]

### 2.6. The Birth and Mortality Rates and Population Dynamics

One of the earlier mathematical models on population and economics is the Haavelmo model (Haavelmo, 1954; Stutzer, 1980). The model considers the following growth of the population

\[ \dot{N}(t) = N(t) \left( a - \frac{\beta N(t)}{Y(t)} \right), \quad a, \beta > 0, \quad Y(t) = AN^\alpha(t), \quad A > 0, \quad 0 < \alpha < 1, \]

where \( N(t) \) is the population, \( Y(t) \) is real output, and \( a, \beta, \alpha \) and \( A \) are parameters. Insert \( Y(t) = AN^\alpha(t) \) in the differential equation

\[ \frac{\dot{N}(t)}{N(t)} = a - \frac{\beta}{f(t)} = a - \frac{\beta N^{1-\alpha}(t)}{A}, \]

where \( f \equiv Y/N \) is per capita output. The Haavelmo model does not consider physical capital accumulation. As the change rate in the population is the birth rate minus the mortality rate, we may interpret that in the Haavelmo model the birth rate \( (= a) \) is constant and the mortality rate \( (= \beta f(t)) \) is negatively related to per capita income. We now mention another approach in the literature of population growth and economic development is the so-called Ramsey model. An example is the model by (Chu et al., 2012; Razin and Ben-Zion, 1975; Yip and Zhang, 1997). The household decides the fertility rate by maximizing the discounted sum of per capita utility across subject to the asset accumulation

\[ U = \int_0^\infty u(c(t), n(t))e^{-rt} dt, \]

s.t.: \[ a(t) = (r(t) - n(t))a(t) + w(t)l(t) - c(t), \]

where \( c(t) \) is the per capita consumption of final goods at time \( t \), \( n(t) \) is the number of births per person, \( a(t) \) is the amount of financial assets per capita, \( r(t) \) is the rate of return on assets, \( w(t) \) is the wage rate, and \( l(t) \) is human capital-embodied labor supply. The total population growth is \( \dot{N} = nN \). As mortality is assumed to be zero in this model, \( n \) is also the growth rate of the population.
Being influenced by the two models just illustrated, we now introduce the population growth. The population change follows

\[ \frac{\dot{N}(t)}{N(t)} = n(t) - d(t), \]  

(14)

where \( n(t) \) and \( d(t) \) are respectively the birth rate and mortality rate. In the Haavelmo model, the mortality rate is negatively related to per capita income. In this study we assume that the mortality rate is negatively related to the disposable income in the following way

\[ d(t) = \frac{\bar{\sigma} N^b(t)}{\bar{y}^b(t)}, \]  

(15)

where \( \bar{\sigma} \geq 0, \) \( b_0 \geq 0. \) We call \( \bar{\sigma} \) the mortality rate parameter. As in the Haavelmo model, an improvement in living conditions implies that people live longer. The term \( N^b(t) \) takes account of possible influences of the population on mortality. For instance, when the population is overpopulated, environment is deteriorated. We may take account of this kind of environmental effects by the term. In this case, it is reasonable to require \( b \) to be positive. It should be noted that the sign of the parameter is generally ambiguous in the sense that the population may also have a positive impact on mortality. Insert (13) and (15) in (14)

\[ \dot{N}(t) = \left( \frac{\nu \bar{\sigma}(t)}{\bar{w}(t)} - \frac{\bar{\sigma} N^b(t)}{\bar{y}^b(t)} \right) N(t). \]  

(16)

The equation describes the population dynamics.

2.7. Wealth Dynamics

According to the definition of \( s(t) \), the change in the household’s wealth is given by

\[ \dot{a}(t) = s(t) - a(t). \]  

(17)

The equation implies that the change in the wealth is the saving minus the dssaving.

2.8. Demand For and Supply of Goods

The national saving is the sum of the households’ saving. As physical capital change is the output of the industrial sector minus consumption of the industrial good and the depreciation of capital stock, we have

\[ K(t) = F(t) - C(t) - \delta K(t), \]  

(18)

where

\[ C(t) = c(t) N(t), \quad K(t) = \bar{k}(t) N(t). \]

2.9. The Gold Owned By the Households

The golds owned by the population is equal to the total fixed gold \( G \)

\[ \bar{g}(t) N(t) = G. \]  

(19)

2.10. Gold Being Fully Used For Decoration

The amount of gold used for decoration by the population is equal to the total gold

\[ \hat{g}(t) N(t) = G. \]  

(20)
2.11. The Value of Physical Wealth and Capital

The value of physical capital is equal to the value of physical wealth

\[ \bar{K}(t)N(t) = K(t). \]  

(21)

We have thus built the dynamic model. The model is general in the sense that the Solow model and the Haavelmo model can be considered as special cases of our model. Since the model in this study is based on the some well-known mathematical models and includes some features which no other single theoretical model explains, we should be able to explain some interactions which other mathematical models fail to explain.

3. THE DYNAMICS AND ITS PROPERTIES

This section examines dynamics of the model. First, we introduce \( z(t) \equiv (r(t) + \delta \dot{z})/w(t) \). We show that the dynamics can be expressed by two differential equations with \( z(t) \) and \( N(t) \) as the variables.

3.1. Lemma

The dynamics of the economic system is governed by the two dimensional differential equations

\[
\begin{align*}
\dot{z}(t) &= \tilde{\Omega}_z(z(t), N(t)), \\
N(t) &= \tilde{\Omega}_N(z(t), N(t)),
\end{align*}
\]  

(22)

where the functions \( \tilde{\Omega}_z \) and \( \tilde{\Omega}_N \) are functions of \( z(t) \) and \( N(t) \) defined in the Appendix. Moreover, all the other variables are determined as functions of \( z(t) \) and \( N(t) \) at any point in time by the following procedure: \( \tilde{k}(t) \) by (A11) \( \rightarrow r(t) \) and \( w_t(t) \) by (A2) \( \rightarrow p_G(t) \) by (A5) \( \rightarrow r_G(t) \) by (4) \( \rightarrow \tilde{g}(t) = \hat{g}(t) \) \( = G/N(t) \) \( \rightarrow a(t) \) by (A15) \( \rightarrow \bar{y}(t) \) by (A6) \( \rightarrow c(t), s(t), \tilde{h}_g(t), \) and \( n(t) \) by (13) \( \rightarrow T_n(t) \) by (9) \( \rightarrow p_G(t) \) by (8) \( \rightarrow T_C(t) \) by (A8) \( \rightarrow \bar{N}(t) \) by (A1) \( \rightarrow K(t) \) by (A1) \( \rightarrow F(t) \) by (2).

The two differential equations (22) contain two variables, \( z(t) \) and \( N(t) \). The appendix shows that the expressions are complicated. It is difficult to explicitly interpret economic implications of the two equations. For illustration, we simulate the model to illustrate behavior of the system. In the remainder of this study, we specify the depreciation rate by \( \delta = 0.05 \), and let \( T_0 = 24 \). We specify the other parameters as follows

\[
\begin{align*}
G &= 1, \quad \alpha = 0.34, \quad \lambda_0 = 0.6, \quad \xi_0 = 0.3, \quad \nu_0 = 0.3, \quad \sigma_{10} = 0.1, \quad \sigma_{20} = 0.1, \quad A = 1, \quad b_0 = 0.4, \\
b &= 0.5, \quad h_1 = 5, \quad h_2 = 4.5, \quad \theta_1 = 2, \quad \theta_2 = 5, \quad \sigma = 1.
\end{align*}
\]  

(22)

The propensity to save is 0.6 and the propensity to consume is 0.3. The total productivity factor is \( A = 1 \). The ratio of the mother’s and father’s time in children fostering 2.5. The father and mother have the same propensity to pursue leisure activities. The father’s human capital is higher than the mother’s. In some empirical studies the value of the parameter, \( \alpha \), in the Cobb-Douglas production is approximately equal to 0.3 (Miles and Scott, 2005). In regard to the preference parameters, what are important in our study are their relative values. To simulate the motion of the system, we specify the initial conditions

\[
\begin{align*}
z(0) &= 0.35, \quad N(0) = 19.
\end{align*}
\]

The simulation result is plotted in Figure 1. The population grows from its low initial condition. As the population rate rises, the mortality rate is also increasing. The labor force is increased and the wage rates are reduced. The falling wage rates reduce the opportunity cost of children fostering, resulting in the
rise of birth rate. The rising in birth rate is associated with rising in both man’s and woman’s time of children fostering. As the income falls, both men and women work longer hours. Their leisure hours are reduced. The national wealth and output are increased in association with rising labor force. Nevertheless, both consumption level and wealth per household are reduced.

It is straightforward to confirm that all the variables become stationary in the long term. This implies the existence of an equilibrium point. The simulation confirms that the system has a unique equilibrium. We list the equilibrium values of the variables as follows

\[
N = 18.67, \quad K = 456.13, \quad \bar{N} = 1580.45, \quad F = 1035.82, \quad n = d = 0.56, \quad r = 0.72,
\]

\[
w_1 = 2.16, \quad w_2 = 1.95, \quad \bar{w} = 114.91, \quad \bar{k} = 100.85, \quad T_1 = 10.40, \quad T_2 = 7.26, \quad \bar{T}_1 = 12.44, \quad \bar{T}_2 = 13.82, \quad \bar{T}_3 = 1.17, \quad \bar{T}_4 = 2.93, \quad c = 50.43.
\]

We calculate the two eigenvalues: \(-0.41\) and \(-0.12\). As the two eigenvalues are negative, the unique equilibrium is locally stable. Hence, the system always approaches its equilibrium if it is not far from the equilibrium.

4. COMPARATIVE DYNAMIC ANALYSIS IN SOME PARAMETERS BY SIMULATION

We simulated the motion of the national economy under (22). As the simulation was conducted under fixed parameters, the results are not robust as we don’t know what happen for other values of the parameters. We now examine how the economic system reacts to some exogenous change. As the lemma gives the computational procedure to calibrate the motion of all the variables with any parameter values, it is straightforward to examine effects of change in a parameter on transitory processes as well stationary states of all the variables. We use a variable \(\Delta x_j(t)\) to stand for the change rate of the variable, \(x_j(t)\), in percentage due to changes in the parameter value.

4.1. A Rise in the Propensity to Use Gold

First, we examine what happen to the economic system when the propensity to use gold is enhanced as follows: \(\gamma_0 \cdot 0.01 \Rightarrow 0.012\). The simulation results are plotted in Figure 2. In order to examine how each variable is affected over time, we should follow the motion of the entire system as each variable is related to the others in complicated interactions. As people desire more strongly to use gold, the gold price and rent become higher. The physical wealth value falls. The birth and death rates are augmented initially and slightly fall in the long term. The population and total labor force are increased. The gold per household falls. The total capital and output level are increased. The rate of interest rises and the wage rates fall. The consumption per household falls initially and rises in the long term. Both the man and woman work longer hours and
reduce leisure hours. The parents spend more hours on fostering children initially and less hours in the long term.

Figure 2. A Rise in the Propensity to Use Gold

Source: Author

4.2. The Propensity to Have Children Being Enhanced

Tournemaine and Luangaram (2012) describe, “depending on the country, population growth may contribute, deter, or even have no impact on economic development. This ambiguous result is explained by the fact that the effects of population growth change over time. For example, a higher fertility rate can have a short-term negative effect caused by the cost of expenditures on children whereas it has a long-run positive effect through the larger labor force it generates.” In order to study transitory processes and long-term results, it is necessary to develop genuine dynamic models. Our model can effectively analyze transitory and long-term issues as we can fully describe the motion of the system for any parameter values. We allow the propensity to have children to be augmented as follows: \( \mu_0 = 0.3 \Rightarrow 0.32 \). The simulation results are plotted in Figure 3. Both the birth and mortality rates are increased. The net result leads to a rise in the population. All the macroeconomic variables real variables are increased. The capital, total labor input and output level are all increased. As the increased population lowers the consumption per capita and wealth per capita, this raises the mortality rate. As the family has more children, the parents spend more time on children caring. Their leisure times are reduced. More children imply less consumption. The rate of interest is increased in association with lowering wage rates. Both the opportunity cost of children \( \tilde{W} \) and the wealth per household are reduced. A stronger desire for children reduces the rent and price of gold initially and enhances the rent and price of gold in the long term. The gold amount owned per household falls.

Figure 3. The Propensity to Have Children Being Enhanced

Source: Author

4.3. The Total Factor Productivity Being Augmented

As the population in the Solow growth model is exogenous, an improvement in the total factor productivity has positive effects on the long-term economic growth. As the Solow model assumes a
constant growth rate and constant return to scales, technological change has no impact on labor inputs and time distribution even in short terms. We now increase the total factor productivity as follows: \( A : 1 \Rightarrow 1.05 \). The output level and wage rate are enhanced. The birth rate is increased initially and is not affected in the long term. The mortality rate is reduced in the short term and is not affected in the long term. The population is augmented and the labor supply is increased. The total capital and output level are increased. The parents spend more time on children caring initially and change the time slightly in the long term. They work more hours and have less leisure time. The wealth per capita and opportunity cost rise over time. The household own less gold. Both the rent and price of gold are enhanced. The consumption level falls

Figure 4. The Total Factor Productivity Being Augmented

Source: Author

4.4. Woman’s Propensity to Pursue Leisure Activities Being Enhanced

Rather than spending more time on bringing up more children, woman may strengthen her preference for pursuing leisure activities. We now allow woman’s propensity to pursue leisure activities to follows: \( \sigma_{o2} : 0.1 \Rightarrow 0.11 \). An immediate consequence of the preference change is that the wife spends more time on leisure and less time on work. The husband has less leisure hours and spends more hours on work. The parents reduce their time of children fostering in the long term. The birth and mortality rates rise initially and fall in the long term. The wage rates rise and rate of interest falls. The population falls. The total labor supply and output level are all reduced. The consumption level, opportunity cost of children fostering, and wealth fall. The amount of gold owned by per household is increased. The rent and price of gold are reduced.

Figure 5. Woman’s Propensity to Pursue Leisure Activities Being Enhanced

Source: Author

4.5. The Propensity to Save Being Increased

According to the Solow model, a rise the propensity to save will increase per capita wealth but reduce per capita consumption level in the long term. The population growth rate is not affected by economic
conditions in this traditional one-sector growth model as the growth rate is exogenous. This study allows the population growth rate to be endogenous. We now allow the propensity to save to rise as follows: \( \lambda_0 : 0.6 \Rightarrow 0.62 \). The family’s wealth and the opportunity cost of children caring are increased. The birth and mortality rates fall. The population falls. The parents spend less time on children. The parents have less leisure time initially and spend a little more time on leisure activities in the long term. They spend more time on work. In the long term the output and total labor supply are reduced. The consumption falls initially and rises in the long term. The amount of gold owned by per household falls. The price of gold falls initially, then rises, and finally falls.

**Figure 6. The Propensity to Save Being Increased**

Source: Author

4.6. **Woman’s Human Capital Being Improved**

The traditional neoclassical approach argues that gender inequalities resulting from disparities in human capital will wither away as an economy experiences high growth (Beneria and Feldman, 1992; Forsythe et al., 2000). According to Stotsky (2006), “the neoclassical approach examines the simultaneous interaction of economic development and the reduction of gender inequalities. It sees the process of economic development leading to the reduction of these inequalities and also inequalities hindering economic development.” We now show how the following rise in the mother’s human capital: \( h_2 : 4.5 \Rightarrow 4.7 \) will affect the dynamic path of the economic system. As the mother accumulates more human capital, her wage is increased. The father’s wage rate is slightly affected. The rate of interest falls. The birth and mortality rates fall and the population rises. The total labor supply, capital stock and output level are all increased. The amount of gold owned by the household falls. The rent and price of gold are increased. The family consumption is increased.

**Figure 7. Woman’s Human Capital Being Improved**

Source: Author
4.7. Father’s Caring Time Per Child Being Increased

We now allow the father to spend more time with each of his children as follows: $\theta_f : 2 \rightarrow 2.4$. The father’s time on children caring is increased and the mother’s time is slightly reduced. The parents spend less time on leisure initially and more hours in the long term. The mother works more hours and the father works less hours. The opportunity cost of children fostering is increased. The family wealth is slightly affected. The rate of interest falls initially and rises in the long term. The total wealth, total labor supply and output level are all reduced. The family consumes less.

Figure 8. Father’s Caring Time per Child Being Increased

Source: Author

5. CONCLUDING REMARKS

This paper introduced endogenous population growth and portfolio equilibrium between physical wealth and gold into the Solow one sector growth model. The study proposed a dynamic interdependence between the birth rate, the mortality rate, the population, wealth accumulation, portfolio equilibrium between physical wealth and gold, and time distribution between work, leisure and children caring. We emphasized the role of human capital, technological and preference changes on the birth and mortality rates, portfolio choice, and time distribution. The approach is based on some well-applied ideas about growth and population change. Gender differences in human capital, the propensity to use leisure time, and children caring efficiency are taken into account in order to explain economic growth. We found a computational procedure to follow the motion of the economic system. As a genuine dynamic analysis is too complicated, we simulated the model. We showed the motion of the economic growth and population change and identified the existence of equilibrium point. We also examined the effects of changes on the transitory processes and long term equilibrium point in the propensity to use gold, the propensity to have children, the propensity to save, woman’s propensity to use leisure, woman’s human capital and man’s emotional involvement in children caring. Our comparative dynamic analysis provides some insights into interdependence between economic growth and population change with portfolio equilibrium. For instance, our simulation demonstrates that if the household’s propensity to use gold for decoration is stronger, the gold price and rent become higher; the physical wealth value falls; the birth and death rates are augmented initially and slightly fall in the long term; the population and total labor force are increased; the gold per household falls; the total capital and output level are increased; the rate of interest rises and the wage rates fall; the consumption per household falls initially and rises in the long term; both the man and woman work longer hours and reduce leisure hours; and the parents spend more hours on fostering children initially and less hours in the long term. There are many ways to generalize and extend our model. Risks are important for model portfolio equilibrium. Another obvious limitation of our model is that children caring function exhibits constant return to scale in the parent’s time spent on children caring. It is possible to generalize our model by applying more general production or utility functions. Our research may also extended and generalized to study some observed phenomena related to gender, human capital and economic development (Stotsky, 2006). It is important to extend our analytical framework to include some of economic forces and population factors. It is also important to generalize the model within a framework of heterogeneous sectors (Jensen and Lehmijoki, 2011; Uzawa, 1961).
Appendix: Proving the Lemma

We now show that the dynamics can be expressed by two differential equations. From (3), we obtain

\[ z \equiv \frac{r + \delta_k}{w} = \frac{\tilde{\alpha} \tilde{N}}{K}, \]  

(A1)

where \( \tilde{\alpha} \equiv \alpha / \beta \). From (2) and (3), we have

\[ r = \alpha A \left( \frac{z}{\tilde{\alpha}} \right)^\beta - \delta_k, \quad w = \beta A \left( \frac{\tilde{\alpha}}{z} \right)^\alpha, \quad w_q = h_q w. \]  

(A2)

We have \( r \), \( w \) and \( w_q \) as functions of \( z \). From the definition of \( \tilde{y} \) and (3)

\[ \tilde{y} = \left(1 + r\right)\alpha + h_0 w, \]  

(A3)

were \( h_0 \equiv \left(h_t + h_2\right)T_0 \). From (13) and (A3) we have

\[ \hat{g} R_s = \left(1 + r\right)\gamma \alpha + \gamma h_0 w. \]  

(A4)

Insert (4) and (5) in (A4)

\[ p_G = \tilde{\gamma} \tilde{k} + \tilde{r}, \]  

(A5)

where we use

\[ \tilde{\gamma}(z, N) = \frac{(1 + r)\gamma}{(r - \gamma - \gamma r)\tilde{g}}, \quad \tilde{\gamma}(z, N) = \frac{\gamma h_0 w}{(r - \gamma - \gamma r)\tilde{g}}. \]

where \( \tilde{g} = G / N \). Insert (5) and (A5) in (A3)

\[ \tilde{y} = \tilde{h} \tilde{k} + \tilde{h}, \]  

(A6)

where

\[ \tilde{h}(z, N) \equiv (1 + \tilde{g} \tilde{\gamma})(1 + r), \quad \tilde{h}(z, N) \equiv (1 + r)\tilde{\gamma} + h_0 w. \]

By (9) and (13), we have

\[ T_q = T_0 - \tilde{T}_q - \tilde{T}_q = T_0 - \left( \frac{\theta_q \nu}{\tilde{w}} + \frac{\sigma_q}{h_q w} \right)\tilde{y}. \]  

(A7)

By the definition of \( \tilde{w} \) we have

\[ \tilde{w} = \tilde{k} + p_G \tilde{g} + hw. \]

Insert (A5) in the above equation

\[ \tilde{w} = \tilde{w}_1 \tilde{k} + \tilde{w}_0, \]  

(A8)
where
\[ \tilde{w}_1(z, N) \equiv 1 + \tilde{\gamma} \tilde{g}, \quad \tilde{w}_2(z, N) \equiv \tilde{\gamma} \tilde{g} + hw. \]

Insert (A6) in (A7)
\[ T_q = \chi_q - (\tilde{h} \tilde{k} + \tilde{h}) \frac{\nu}{\tilde{w}} - \tilde{r}_q \tilde{k}, \]  \hspace{1cm} (A9)

where
\[ \chi_q(z, N) \equiv T_q - \frac{\sigma_q \tilde{h}}{h_q w}, \quad \tilde{r}_q(z, N) \equiv \frac{\sigma_q \tilde{h}}{h_q w}. \]

Insert (A9) in (1)
\[ \frac{N}{N} = \chi - \tilde{r} \tilde{k} - \frac{(\tilde{h} \tilde{k} + \tilde{h}) \nu}{\tilde{w}}, \]  \hspace{1cm} (A10)

where
\[ \chi(z, N) \equiv h_1 \chi_1 + h_2 \chi_2, \quad \tilde{r}(z, N) \equiv h_1 \tilde{r}_1 + h_2 \tilde{r}_2, \quad \tilde{h}_0 \equiv h_1 \theta_1 + h_2 \theta_2. \]

Insert \( kN = K \) in (A1)
\[ z = \tilde{\alpha} \frac{N}{kN}. \]  \hspace{1cm} (A11)

Insert (A10) in (A11)
\[ \left( \frac{z}{\tilde{\alpha}} + \tilde{r} \right) \tilde{k} = \chi - \frac{(\tilde{h} \tilde{k} + \tilde{h}) \nu}{\tilde{w}}, \]  \hspace{1cm} (A12)

Insert (A8) in (A12)
\[ \left( \frac{z}{\tilde{\alpha}} + \tilde{r} \right) \tilde{k} = \chi - \frac{\nu}{\tilde{w}} \left( \tilde{h} \tilde{k} + \tilde{h}_0 \nu \right), \]  \hspace{1cm} (A13)

We rewrite (A13) as
\[ \tilde{k}^2 + \tilde{m}_1 \tilde{k} + \tilde{m}_2 = 0, \]  \hspace{1cm} (A13)

where
\[ \tilde{m}_1(z, N) \equiv \left( \frac{\tilde{w}_0 \frac{z}{\tilde{\alpha}} + \tilde{w}_0 \tilde{r} + \tilde{h} \tilde{h}_0 \nu - \chi \tilde{w}_1}{\tilde{m}} \right) \frac{1}{\tilde{m}}, \]
\[ \tilde{m}_2(z, N) \equiv \frac{\tilde{h}_0 \nu}{\tilde{m}} - \chi \frac{\tilde{w}_0}{\tilde{m}}, \quad \tilde{m}(z, N) \equiv \left( \frac{z}{\tilde{\alpha}} + \tilde{r} \right) \tilde{w}_1. \]
Solve (A13) with $\bar{k}$ as variable

$$\bar{k}(z, N) = \frac{-\bar{m}_1 \pm \sqrt{\bar{m}_1^2 - 4\bar{m}_2}}{2}.$$  \hspace{1cm} (A14)

From (5) we have

$$a = \Lambda(z, N) = \bar{k} + p_G \bar{g}.$$ \hspace{1cm} (A15)

From (A15) we solve $a$ as a function of $z$ and $N$. The following procedure shows how to express the variables as functions $z$ and $N$: $\bar{k}$ by (A11) $\rightarrow$ $r$ and $w_q$ by (A2) $\rightarrow$ $p_G$ by (A5) $\rightarrow$ $r_q$ by (4) $\rightarrow$ $\bar{g} = \hat{g} = G/N \rightarrow$ $a$ (A15) $\rightarrow$ $\bar{N}$ by (A9) $\rightarrow$ $\bar{y}$ by (A6) $\rightarrow$ $c$, $s$, $T_q$, and $n$ by (13) $\rightarrow$ $\bar{a}$ by (9) $\rightarrow$ $p_b$ by (8) $\rightarrow$ $T_q$ by (A8) $\rightarrow$ $\bar{N}$ by (A1) $\rightarrow$ $K$ by (A1) $\rightarrow$ $F$ by (2). From this procedure and (14), it is straightforward to show that the motion of the population can be expressed as a function of $z(t)$ and $N(t)$ at any point in time

$$N(t) = \Omega_N(z(t), N(t)).$$ \hspace{1cm} (A16)

We now show that change in $z(t)$ can also be expressed as differential equations with $z(t)$ and $N(t)$ as variables. First we note that both $\bar{k}$ and $\bar{y}$ can be expressed as functions of $z(t)$ and $N(t)$. From (17), we have

$$\dot{a} = \Omega_\delta(z, N) = \lambda \bar{y} - a.$$ \hspace{1cm} (A17)

Take derivatives of (A17) with regard to $t$

$$\dot{a} = \frac{\partial \Lambda}{\partial z} \dot{z} + \frac{\partial \Lambda}{\partial N} \dot{N}.$$ \hspace{1cm} (A18)

Equal (A17) and (A18)

$$\dot{z} = \Omega_{\delta}(z, N) = \left(\Omega_\delta - \frac{\partial \Lambda}{\partial N} \Omega_N \right) \left(\frac{\partial \Lambda}{\partial z}\right)^{-1},$$ \hspace{1cm} (A19)

where we also use (A16). In summary we proved the lemma.

REFERENCES


Zhang, W. B. (2016). Gold and land prices with capital accumulation in an economy with industrial and agricultural sectors. Annals - Economy Series; Constantin Brancusi University.